1.

\[
\begin{array}{cccc}
& t_1 & t_2 & t_3 & t_4 \\
1 & \begin{array}{cc}
S_1 & 2, 2 \\
S_2 & 0, x \\
S_3 & 0, 0 \\
S_4 & 0, -1
\end{array} & \begin{array}{cc}
x, 0 \\
4, 4 \\
0, 0 \\
0, -1
\end{array} & \begin{array}{cc}
-1, 0 \\
-1, 0 \\
0, 2 \\
-1, -1
\end{array} & \begin{array}{cc}
0, 0 \\
0, 0 \\
0, 0 \\
2, 0
\end{array}
\end{array}
\]

Note that \((s_i, t_i)\) for \(i = 1, 3,\) and 4 are all Nash equilibria in the stage game; so the punishments and rewards in stage 2 are all credible. Since the payoffs are symmetric the same calculation works for both players: \(4 + 2 \geq x + 0,\) so \(x \leq 6.\)
2.

<table>
<thead>
<tr>
<th></th>
<th>t₁</th>
<th>t₂</th>
<th>t₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>s₁</td>
<td>1, 1</td>
<td>5, 0</td>
<td>0, 0</td>
</tr>
<tr>
<td>s₂</td>
<td>0, 5</td>
<td>4, 4</td>
<td>0, 1</td>
</tr>
<tr>
<td>s₃</td>
<td>0, 0</td>
<td>1, 0</td>
<td>-1,-1</td>
</tr>
</tbody>
</table>

1 1 0 = 2’s minmax strategy

Supporting (4,4) requires δ and N that satisfy:

(i) \[ 4 > 5(1 - \delta) + \delta p^* \], and
(ii) \[ p^* = (1 - \delta^N)(-1) + \delta^N(4) > 0. \]

At \( \delta = .9 \) and \( N = 1 \), we have \( p^* = 4(.9) - .1 = 3.5 \) and \( .1(5) + .9(3.5) = 3.65 < 4 \), as required. One period of punishment is needed.

As before to support (3/4, 3/4) requires δ and N that satisfy:

(i) \[ 3/4 > .5 + .9p^* \], and
(ii) \[ p^* = .9^N(3/4) - 1 + .9^N > 0. \]

At \( N = 1 \), the punishment payoff \( p^* = .575 > 0 \), but \( 3/4 < .5 + .9(.575) = 1.02 \), so (i) is violated. At \( N = 2 \), \( p^* = .4175 > 0 \), but \( 3/4 < .5 + .9(.4175) = .876 \) so (i) is violated again. However, if we set \( N = 3 \), then \( p^* = .2757 > 0 \) and \( 3/4 > .5 + .9(.2757) = .74818 \). Hence, a three-period punishment phase is needed to support (3/4, 3/4).
3. We begin with the case that \( \delta = 0.85 \) and \( K = 4 \). Note that \( \delta^2 < \frac{3}{4} \). Denote the players making the first and second moves P1 and P2, respectively. We use \((p, 1-p)\) to denote a division that gives P1 a fraction \( p \) of the item and P2 a fraction \( 1-p \).

(a) In round 4 (the last round), the best responses are for P1 to accept any offer, and for P2 to offer \((0, 1)\).

In round 3, the best responses are for P2 to accept any offer giving them at least \( \delta \). Since the only offer satisfying this is \((0, 1)\), every possible offer yields P1 a value of 0 and so any offer is a best response.

In round 2, we have two cases depending on what offer P1 makes in round 3:

1. If P1 offers \((0, 1)\) in round 3, then P2’s best response is to offer \((0, 1)\), since any other offer yields less than \( \delta \), the discounted value of accepting the offer P1 makes in round 3. On the other hand, P1’s best response is to accept any offer, since rejecting yields value 0 as well.

2. If P1 makes any offer besides \((0, 1)\) in round 3 (leading P2 to reject the offer), then P2’s best response is to offer \((\frac{3}{4}, \frac{1}{4})\) since \( \frac{3}{4} > \delta^2 > \frac{1}{2} \). P1’s best response is to accept any offer yielding at least \( \frac{1}{4} \) value.

In round 1, we again have two cases, depending on what offer P2 makes in round 2:

1. If P2 offers \((0, 1)\) in round 2, then the payoffs are identical to round 3, and the same best responses hold;

2. If P2 offers \((\frac{1}{4}, \frac{3}{4})\) in round 2, then P2’s best response in round 1 is to only accept offers yielding at least \( \frac{3}{4} \delta \) value (i.e. \((0,1)\) or \((1/4, 3/4)\)). P1’s best response is to offer \((1/4, 3/4)\).

Thus, equilibrium strategies are:
P1: \((\text{Offer } \frac{3}{4}, \text{Accept } \geq \frac{1}{4}, \text{Offer } \leq \frac{3}{4}, \text{Accept } \geq 0)\); P2: \((\text{Accept } \geq \frac{3}{4}, \text{Offer } \frac{1}{4}, \text{Accept } \geq 1, \text{Offer } 0)\)
P1: \((\text{Offer }, \text{Accept } \geq 0, \text{Offer } 1, \text{Accept } \geq 0)\); P2: \((\text{Accept } \geq 1, \text{Offer } 0, \text{Accept } \geq 1, \text{Offer } 0)\)

(b) In the infinite-horizon version of the problem, there are subgame perfect equilibria supporting every possible division: \(\{\left((0,1), \left(\frac{1}{4}, \frac{3}{4}\right), \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{3}{4}, \frac{1}{4}\right), (1,0)\}\}\). In particular, the division \(\left(\frac{k}{4}, 1 - \frac{k}{4}\right)\) is supported by the strategies:
P1: \((\text{Offer } 1 - \frac{k}{4}, \text{Accept } \geq \frac{k}{4}, \text{Offer } 1 - \frac{k}{4}, \text{Accept } \geq \frac{k}{4}, \ldots)\)
P2: \((\text{Accept } \geq 1 - \frac{k}{4}, \text{Offer } \frac{k}{4}, \text{Accept } \geq 1 - \frac{k}{4}, \text{Offer } \frac{k}{4}, \ldots)\)

We can see these strategies are subgame perfect by applying the one-shot deviation principle. First, we can see that no agent making an offer can benefit from deviating: an offer of more will still be accepted, reducing their own payoff; and an offer of less will be rejected, yielding them the same payoff in the next round, but with a discount factor applied. Second, we can see that no agent receiving an offer can benefit from deviating: the only deviation that affects payoffs is to change to rejecting a previously received offer, but again, this just produces the same payoff next round but with a discount factor applied.

(c) Revisiting (a), we consider how increasing \( T \) will affect the set of equilibria. Consider what happens when we increase the number of rounds to \( T = 6 \). So here, round 3 will look like round 1 from (a). First, observe that if we are in the case where P1 makes an arbitrary offer in round 3
and an offer of (0, 1) in round 5, rounds 1 and 2 for \( T' = 6 \) will follow the same logic as we followed for rounds 1 and 2 in the case \( T' = 4 \).

On the other hand, in the case where P1 makes an offer of (1/4, 3/4) in both rounds 3 and 5, rounds 1 and 2 will look exactly like rounds 3 and 4, respectively, from the perspective of both agents. Observe that we have reached a fixed point in the backward induction, and so whichever player has the last move can achieve payoff no more than \( 1/4 \) when \( \delta = 0.85 \) and \( K = 4 \).

Clearly, the set of divisions supported in the finite-horizon case does not converge to the set of divisions supported in the infinite-horizon case as the number of rounds \( T' \to \infty \).

One intuition for this is the following: the expanded range of divisions in the case of a finitely-divisible item arises from the fact that many threats which were incredible (and hence eliminated in SPE) can become credible when the other player’s ability to make counter-offers is limited to a restricted set. One the other hand, in the finite-horizon case, the player with the last move can keep the entire item if the last round is ever reached. This provides extra avenues for removing threats in subgames, refining the set of possible divisions by reducing the set of credible threats.