Problem Set 3: Suggested Answers

1. While the specific problem given can be solved with a simplified version of the following proof, we use the approach below to show how more general settings would be approached. Consider an n-bidder auction with valuations \( v_i \) independently and identically distributed according to \( F(v_i) \) on support \([v, v]\). Let the highest bidder pay the price \((1 - k)b_f + kb_s\) to the seller, where \( k \in [0, 1] \), \( b_f = \) the (first) highest of the n bids, and \( b_s = \) the second-highest of the n bids. This is a \( k+1 \) price auction: if \( k = 0 \) we have a first-price auction, and if \( k = 1 \) we have a second-price auction. The utility function for bidder i is then

\[
u_i(b, b, v, v) = \begin{cases} v_i - [(1 - k)b_i + k \cdot \max_{j \neq i} b_j] & \text{if } b_i > \max_{j \neq i} b_j, \\ 0 & \text{otherwise.} \end{cases}
\]

So if the \( n - 1 \) other bidders play the strategy \( b(v_j) \), i's best response solves

\[
\max_{b_i} \int_0^{b^{-1}(b_i)} v_i - (1 - k)b_i + kb(y) \, dG(y)
\]

where \( y = \max_{j \neq i} v_j \) and \( G(y) = F^{n-1}(y) \) is the distribution of the highest of \( n - 1 \) random variables independently distributed according to \( F(\cdot) \). The first-order condition is

\[
0 = \int_0^{b^{-1}(b_i)} (1 - k)dG(y) + \frac{d}{db_i} b^{-1}(b_i) \left[ v_i - [(1 - k)b_i + kb(b^{-1}(b_i))] \right] g[b^{-1}(b_i)]
\]

where \( g(y) = (n - 1)F^{n-2}(y)f(y) \) is the density of \( G \) and \( f \) is the density of \( F \). Using the fact that

\[
\frac{d}{db_i} b^{-1}(b_i) = \frac{1}{b'[b^{-1}(b_i)]},
\]

The first-order-condition becomes

\[(*) \quad (1 - k)b'[b^{-1}(b_i)] = (v_i - b_i)(n - 1) = \frac{f[b^{-1}(b_i)]}{F[b^{-1}(b_i)]}\]

For the second-price auction \( (k = 1) \), \((*)\) implies \( b_i = v_i \) \textit{no matter what} strategy \( b(\cdot) \) the others are playing. In fact, it is a dominant strategy for each player to bid \( b_i = v_i \). For any other \( k \in [0, 1) \), \((*)\) determines a symmetric equilibrium provided its solution \( b_i \) is in fact the hypothesized \( b(v_i) \). So writing \( v_i \) for \( b^{-1}(b_i) \) yields a linear first-order differential equation

\[
b'(v_i) = \frac{n-1}{1 - k} \frac{f(v_i)}{F(v_i)} [v_i - b(v_i)],
\]

which can be solved using an integrating factor.

A short review on integrating factors: Suppose we wish to solve
\[ y'(x) + p(x)y(x) = q(x). \]

Define

\[ P(x) = \int_a^x p(z) \, dz. \]

Then

\[ e^{P(x)} \{ y'(x) + P(x)y(x) \} = \frac{d}{dx} \{ e^{P(x)}y(x) \}, \]

so

\[ y(x) = e^{-P(x)} \int e^{P(x)}q(x) \, dx. \]

Proceeding analogously, letting \( m = (n - 1)/(1 - k) \), yields

\[ P(v_i) = \int m \frac{f(z)}{F(z)} \, dz = m \ln \{ F(v_i) \}, \]

so the integrating factor is \([F(v_i)]^m\). Then

\[ b(v_i) = [F(v_i)]^{-1} \int v_i \frac{d}{dv_i} [F(v_i)^m]dFv_i \]

\[ = [F(v_i)]^{-1} \left\{ v_i F(v_i)^m - \int_a^{v_i} F(z)^m \, dz + c \right\} \]

\[ = v_i - \frac{1}{F(v_i)^m} \int_a^{v_i} F(z)^m \, dz, \]

where the values of \( a \) and \( c \) must satisfy

\[ c = -\int_a^x F(z)^{m-1} \, dz, \]

since otherwise \(|b(v_i)| \to \infty \) as \( v_i \to \lambda \).

For the assigned setting, we have \( k = 0 \), \( n = 3 \), and \( F(z) = z \), and so this equilibrium strategy becomes

\[ b(v_i) = 2v_i/3. \]

2. The payoff functions for this game are

\[ u_i(b_i, b_{-i}, v_i, v_{-i}) = \begin{cases} 
 b_i - v_i & \text{if } b_i < \min_{j \neq i} b_j, \\
 \frac{1}{n-1} \min_{j \neq i} b_j & \text{otherwise.} 
\end{cases} \]

If the \( n-1 \) others are playing \( b(\cdot) \) then \( i \)'s best response given value \( v_i \) solves
\[
\max_{b_i} \int_{b^{-1}(b_i)}^{v_i} (b_i - v_i) dH(z) + \int_{V}^{b^{-1}(b_i)} - \frac{1}{n-1} b(z) dH(z)
\]

where

\[
H(z) = \Pr\{\min_{j \neq i} v_j < z\} = 1 - \Pr\{\min_{j \neq i} v_j \geq z\} = 1 - [1 - F(z)]^{n-1}.
\]

The first-order condition is

\[
0 = \int_{b^{-1}(b_i)}^{v_i} dH(z) - \frac{d}{db_i} - \frac{1}{n-1} b(z) dH(z) = \frac{b_i - v_i}{n} \left\{ \frac{d}{db_i} \left( (b_i - v_i) h(b^{-1}(b_i)) + \frac{1}{n-1} b(b^{-1}(b_i)) h(b^{-1}(b_i)) \right) \right\},
\]

where \( h(z) = H'(z) \). By the usual series of steps, this condition becomes

\[
b'(v_i) = \frac{h(v_i)}{1 - H(v_i)} \left( \frac{n}{n-1} b(v_i) - v_i \right) = (n-1) \frac{f(v_i)}{1 - F(v_i)} \left( \frac{n}{n-1} b(v_i) - v_i \right).
\]

Using an integrating factor as above yields

\[
b(v_i) = \frac{n-1}{n} \left\{ v_i + \int_{v_i}^{v} \left[ 1 - F(z) \right]^n dz \right\},
\]

where the constants in the indefinite integral are determined the same way as above. For the uniform case, \( F(z) = z \), so we get

\[
b(v_i) = \frac{n-1}{n+1} (v_i + 1/n).
\]

3. (a) There are two pure-strategy equilibria: player 1 always raises, so player 2 quits immediately, yielding a payoff of (9, 0); and player 2 always raises, so player 1 quits immediately, yielding a payoff of (0, 10).

(b) Past bids should be thought of as sunk costs, since they are paid regardless of whether you win or lose. Note that, except for the initial bid, we have a stationary game like in Rubinstein's bargaining problem: the game at any point in the future looks the same—you pay $2 each period for the right to continue your chance of winning the $10 prize. Consider the decision problem faced by player 1 after an arbitrary number of periods, but not the initial decision. For player 1 to randomize between raising and quitting, she must be indifferent between each of her pure strategies \{Q, RQ, RRQ, RRRQ, ...\} where, for example, RRQ denotes raising in the next two periods, but quitting in the third. The payoffs for each of these strategies are shown below:
\[ \begin{array}{|c|c|} \hline \text{Pure Strategy} & \text{Payoff (ignoring sunk costs)} \\ \hline Q & 0 \\ RQ & -2 + 10q \\ RRQ & -2 + 10q + (1 - q)(-2 + 10q) = (-2 + 10q)(1 + 1 - q) \\ RRRQ & (-2 + 10q)[1 + 1 - q + (1 + q)^2] \\ \hline \end{array} \]

Clearly, the unique value of \( q \) that makes player 1 indifferent among each of these pure strategies is \( q = .2 \).

A similar analysis for player 2 results in \( p = .2 \). Indeed, this is a unique mixed strategy equilibrium, since indifference between \( Q \) and \( RQ \) (a necessary condition for randomization) requires \( p = q = .2 \). Hence, quitting with probability .2 at each information set forms a Nash equilibrium. It is subgame-perfect, since each information set is reached with positive probability (we never get off the equilibrium path).

Moreover, we have shown that the randomization is a Nash equilibrium at an arbitrary information set.]

It remains to show that player 1’s best response is to bid \$1 in the initial period. By quitting 1 gets zero, whereas bidding \$1 leads to an expected payoff of \(-1 + (.2)(10) + (.8)(0) = \$1\), regardless of which pure strategy 1 employs. Hence, player 1 should bid \$1 with probability one.

(c) The first-mover advantage in this game is \$1, since the expected payoff to player 1 is \$1 whereas the expected payoff to player 2 is \$0. Player 1 would be willing to pay up to \$1 for the right to bid first.

(d) Part (b) shows that if the players are randomizing, then they must be randomizing with probability \(.2\). Suppose at some information set, player i quits with probability one. Then at the preceding node, player j must raise with probability one, and by backward induction, we get an equilibrium equivalent to that in (a). On the other hand, suppose that at some information set, player i raises with probability one. Then player j must quit at the preceding node, for otherwise we can get a contradiction: either i raising is not a best-response or j raising is not a best-response. But this also leads to the equilibrium in (a).