Chapter 3

Nash-Equilibrium for Two-Person Games

Zero-sum Games and Constant-sum Games

- Definition of zero-sum games
  - Examples: Poker, Battle of the Networks
- The arrow diagram for a $2 \times 2$ game in normal form
  - The arrows point towards a Nash equilibrium
- Transforming a constant-sum game into a zero-sum game
### Battle of the Networks, normal form: The payoff matrix

<table>
<thead>
<tr>
<th>Network 1</th>
<th>Sitcom</th>
<th>Sports</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sitcom</td>
<td>55%, 45%</td>
<td>52%, 48%</td>
</tr>
<tr>
<td>Sports</td>
<td>50%, 50%</td>
<td>45%, 55%</td>
</tr>
</tbody>
</table>

Network 1’s Best Responses are underlined.

### Battle of the Networks, normal form: Strategy for Network 1

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Network 1’s Best Responses are underlined.
Battle of the Networks, normal form: Strategy for Network 2

Network 2

Network 1

Sitcom

Sports

55%, 45%  
52%, 48%

Network 2’s Best Responses are underlined.

45%, 55%

Battle of the Networks, normal form: The Equilibrium

Network 2

Network 1

Sitcom

Sports

55%, 45%  
52%, 48%

All Best Responses are underlined.

50%, 50%  
45%, 55%
Battle of the Networks, zero-sum form: The payoff matrix

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<td><strong>Network 1</strong></td>
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<td></td>
</tr>
<tr>
<td>Sitcom</td>
<td>10%, -10%</td>
<td>4%, -4%</td>
</tr>
<tr>
<td>Sports</td>
<td>0, 0</td>
<td>-10%, 10%</td>
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</tbody>
</table>

Network 1’s Best Responses are underlined.

Battle of the Networks, zero-sum form: Strategy for Network 1

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</tr>
<tr>
<td>Sports</td>
<td>0, 0</td>
<td>-10%, 10%</td>
</tr>
</tbody>
</table>
Battle of the Networks, zero-sum form: Strategy for Network 2

Network 2

Network 1

Sitcom Sports

Sitcom

10%, -10% 4%, -4%

Sports

0, 0 -10%, 10%

Network 2’s Best Responses are underlined.

Battle of the Networks, zero-sum form: The equilibrium

Network 2

Network 1

Sitcom Sports

Sitcom

10%, -10% 4%, -4%

Sports

0, 0 -10%, 10%

All Best Responses are underlined.
Why look for Nash equilibrium?

- Equilibrium concept
  - No player has incentive to change strategy
- Analogy to dominance solvability
  - Play of strictly dominant strategies leads to Nash equilibrium
  - Weak dominance solution is also Nash equilibrium

Nash equilibrium of a game like chess

a) Extensive Form

b) Normal Form
Competitive Advantage

- The game Competitive Advantage and the economic realities it reflects
- The solution of Competitive Advantage, showing why firms are driven to adopt new technologies

Competitive Advantage: The payoff matrix

<table>
<thead>
<tr>
<th></th>
<th>New Technology</th>
<th>Stay Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>Firm 1 Stay Put</td>
<td>-a , a</td>
<td>0 , 0</td>
</tr>
<tr>
<td>New Technology</td>
<td>a , -a</td>
<td>-a , a</td>
</tr>
<tr>
<td>Firm 2 Stay Put</td>
<td>a , -a</td>
<td>0 , 0</td>
</tr>
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</table>
Competitive Advantage: Strategy for Firm 1

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<td>0, 0</td>
<td>a, -a</td>
</tr>
<tr>
<td>New Technology</td>
<td>-a, a</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

Competitive Advantage: Strategy for Firm 2

<table>
<thead>
<tr>
<th></th>
<th>New Technology</th>
<th>Stay Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>Firm 2, New Technology</td>
<td>0, 0</td>
<td>a, -a</td>
</tr>
<tr>
<td>Firm 1, Stay Put</td>
<td>-a, a</td>
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1-Card Stud Poker

- A model of a more complicated game, made simpler by fewer kinds of cards and smaller hands
- How imperfect information is created by the deal of the cards
- Principles of Poker reflected by the solution of 1-card Stud
- The concept of dominated strategy
Stud Poker, the rules

1. Each player places 1 chip into the pot
2. Each player is dealt 1 card, face-down
   1. Deck has 50% Aces and 50% Kings
   2. No one looks at their card
3. Each player decides whether to make a further bet of one more chip or not. A player does this by placing 1 chip in hand behind his or her back (or not), out of sight of the other player.
4. Each player simultaneously reveals further bet.
5. If only one player has made a further bet, then that player wins the pot and no cards need be shown.
6. If both players have made a further bet of either 1 or 0 chips, then there is a showdown. High card wins the pot; in case of a tie, the players split the pot.

One-Card Stud Poker, extensive form

```
0
1/4 (A, A)
1/4 (A, K)
1/4 (K, A)
1/4 (K, K)
1
Bet
Pass
Bet
Pass
Bet
Pass
Bet
Pass

(0, 0)
(a, -a)
(-a, a)
(0, 0)
(a+b, -a-b)
(a, -a)
(-a, a)
(a, -a)
(-a-b, a+b)
(a, -a)
(-a, a)
(-a, a)
(0, 0)
(a, -a)
(-a, a)
(0, 0)
```
One-Card Stud Poker, normal form: The payoff matrix

\[
\begin{array}{c|c|c|c}
 & \text{Bet} & \text{Pass} \\
\hline
\text{Bet} & 0, 0 & a, -a \\
\hline
\text{Pass} & -a, a & 0, 0 \\
\end{array}
\]

One-Card Stud Poker, normal form: Strategy for player 1

\[
\begin{array}{c|c|c|c}
 & \text{Bet} & \text{Pass} \\
\hline
\text{Bet} & 0, 0 & a, -a \\
\hline
\text{Pass} & -a, a & 0, 0 \\
\end{array}
\]
One-Card Stud Poker, normal form:
Strategy for player 2

One-Card Stud Poker, normal form:
The equilibrium
Nash Equilibria of 2-Player, 0-Sum Games

- Two examples of 2-player, 0-sum games with multiple Nash equilibria
- Every equilibrium of a 2-player, 0-sum game has the same payoffs (proved by contradiction for the 2×2 case)

Two-person, zero-sum game with two solutions: The payoff matrix

```
Player 2

Player 1

<table>
<thead>
<tr>
<th></th>
<th>Left</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Left</td>
<td>1, -1</td>
<td>0, 0</td>
</tr>
<tr>
<td>Right</td>
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Two-person, zero-sum game with two solutions: Strategy for player 1

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Two-person, zero-sum game with two solutions: Strategy for player 2

Player 2

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Two-person, zero-sum game with two solutions: Two equilibria

Two-person, zero-sum game with four solutions: The payoff matrix
Two-person, zero-sum game with four solutions: Strategy for Player 1

Player 1

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<thead>
<tr>
<th></th>
<th>Left</th>
<th>Center</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Left</td>
<td>0, 0</td>
<td>1, -1</td>
<td>0, 0</td>
</tr>
<tr>
<td>Center</td>
<td>-1, 1</td>
<td>0, 0</td>
<td>-1, 1</td>
</tr>
<tr>
<td>Right</td>
<td>0, 0</td>
<td>1, -1</td>
<td>0, 0</td>
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Two-person, zero-sum game with four solutions: Strategy for Player 2

Player 1

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Two-person, zero-sum game with four solutions: Four equilibria

Von Neumann’s Theorem for 2-Player Zero-Sum Games

- Every equilibrium of a 2-player zero-sum game has the same payoffs
- No benefit to cooperation
Proof of van Neumann’s theorem: By contradiction, suppose $a > b$

2-Player Variable Sum Games

- The concept of a variable sum game
- Most games in business are variable sum
- A variable sum game may have equilibria with different payoffs
- Necessary and sufficient conditions for a solution
- The sufficient condition of undominated strategies
- Games as parables
Let’s Make a Deal: The payoff matrix

![Payoff Matrix Diagram]

Let’s Make a Deal: Strategy for the Movie Star

![Strategy Diagram]
Let’s Make a Deal:
Strategy for the Director

<table>
<thead>
<tr>
<th></th>
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<th>No</th>
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<tr>
<td>Yes</td>
<td>$15M, $15M</td>
<td>0, 0</td>
</tr>
<tr>
<td>No</td>
<td>0, 0</td>
<td>0, 0</td>
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Let’s Make a Deal: Two equilibria

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Video System Coordination:  
The payoff matrix

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<th>VHS</th>
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<td>Firm 2</td>
<td>1, 1</td>
<td>0, 0</td>
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Video System Coordination: Strategy for Firm 2

The two equilibria
**Prisoner’s Dilemma:**

**The payoff matrix**

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<th>Confess</th>
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<td>-2, 2</td>
</tr>
<tr>
<td>Confess</td>
<td>2, -2</td>
<td>-1, -1</td>
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All numbers are years of time served

**Prisoner’s Dilemma:**

**Strategy for Player 1**

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**Prisoner’s Dilemma: The equilibrium**

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47

48
Cigarette Advertising on Television

- Advertising as a strategic variable
- Heavy advertising by each firm as a dominant strategy and a game equilibrium
- How the ban on cigarette advertising on television in 1971 raised industry profits
- The prisoner’s Dilemma

Cigarette Television Advertising: The payoff matrix

<table>
<thead>
<tr>
<th>Company 2</th>
<th>Don’t advertise on television</th>
<th>Advertise on television</th>
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<tbody>
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<td>Company 1</td>
<td>$50M, $50M</td>
<td>$20M, $60M</td>
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<td>$60M, $20M</td>
<td>$27M, $27M</td>
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Cigarette Television Advertising: Strategy for Company 1

Company 2

Company 1

Don’t advertise on television

$50M, $50M

$20M, $60M

Advertise on television

$60M, $20M

$27M, $27M

Cigarette Television Advertising: Strategy for Company 2

Company 2

Company 1

Don’t advertise on television

$50M, $50M

$20M, $60M

Advertise on television

$60M, $20M

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2-Player Games with Many Strategies

- Using calculus to maximize utility and solve games
- Equilibrium as the solution of a pair of first order conditions
- Advertising budgets as continuous strategic variables
Game with many strategies, the corners

Existence of Equilibrium

- The best response function
- A fixed point theorem
- The Existence of Equilibrium
Strictly dominance solvable game, best response function

\[ f_1(x_2) \]

\[ f_2(x_1) \]

Nash Equilibrium

\[ (500, 250) \]

Strictly dominance solvable game, best response mapping
Advertising, many strategies, best response functions

\[ f_1(x_2) \]

\[ X^* = (500, 250) \]

\[ f_2(x_1) \]

\[ x_1 \]

\[ x_2 \]

Advertising, many strategies, best response functions

\[ f(x^*) \]

\[ X^* = (500, 250) \]

\[ f^*(x) \]

\[ x_1 \]

\[ x_2 \]